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Transition temperatures of the trapped ideal spinor Bose gas

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Abstract. The thermodynamic properties of the trapped ideal spinor Bose gas are studied in details with the constraints of fixed total number of atoms N, and magnetization M. The double transition temperatures, their corresponding corrections due to finite particle number, and the population of each component are investigated. The generalization to the ideal spinor Bose gas of hyperfine quantum number F is also discussed. We propose that the order and disorder parameters to describe the symmetry broken of condensation.

PACS. 03.75.Mn Multicomponent condensates; spinor condensates – 03.75.Hh Static properties of condensates; thermodynamical, statistical, and structural properties – 05.30.Jp Boson systems (for static and dynamic properties of Bose-Einstein condensates)

1 Introduction

Since the experimental realization of Bose-Einstein condensation (BEC) in 1995 [1], there are two kinds of atomic trapping. Namely, the *magnetic optical trap* (MOT), and the *all-optical trap* (AOT) [2]. In the AOT, the atoms are confined by the dipole potential of the light. For atoms with hyperfine quantum number F, all the $2F + 1$ components might be trapped. The condensate is called the spinor BEC. Due to the coexistence of multi-component condensates, the spinor system has interesting property that MOT BEC does not have. For example, due to the difference in atomic scattering lengths, the $F = 1$ spinor BEC of ^{87}Rb atom is a ferromagnetic state and ^{23}Na is a polar state [3]. The magnetic property is involved. Theoretical studies on spinor BEC started by Ohmi and Machida [4], Ho [5], Yip [6], Huang and Gou [7] etc. In several earlier works, the conservation of magnetization was not considered. Yi et al. [8] showed the ground state without the constraint of conserved M will generally deviate from the true ground state with the condition of M.

On the other hand, the statistical mechanical properties of spinor system were less explored than the MOT type BEC. Isoshima et al. [9] found the double phase transitions of ideal spinor Bose gas. They considered the conservation of magnetization but the effect of finite particle number was not studied. Huang et al. studied analytically the weakly interacting effect on the transition temperature [10] under applied magnetic field, but the constraint of the magnetization M conservation was not considered. Recently, Zhang et al. developed a Hartree-Fock-Popov type approximation to study the behavior of spinor BEC [11]. They pointed out that there may have phase transitions more than just double phase transitions. However, the effects of finite particle number are also left over.

In this paper, we will focus on the statistical mechanical properties of the $F = 1$ spinor BEC. This is applicable to current spinor BEC systems of ^{87}Rb and ^{23}Na. We will take both the constraints of finite particle number and magnetization into considerations. And derive the corrections due to the finite particle number. At this moment, we present the results on ideal gas only. For the interacting dilute Bose gas, we had studied the transition temperature of the single component BEC [12]. We found that near the transition temperature, both the condensate and the thermal gas affect the statistical properties, the effect of condensate part on the transition temperature was not mentioned before [13]. Our results agree with the experiment very well [14]. On the other hand, the statistical mechanics of dilute spinor Bose gas with interaction is quite complicated. And we found that the analysis of statistical mechanics of the current spinor Bose gas is still incomplete. Thus, in this paper, we provide a detailed study for the transition temperatures of the ideal spinor Bose gas based on the method mainly developed by Pathria [15]. The basic thermodynamic properties, including the double transition temperatures, the effect of finite particle number on the transition temperature and population of each component are investigated. The implications of weakly interacting spinor Bose gas will be presented in future work.

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The layout of this paper is as follows. In Section 2, the basic formulation of the ideal $F = 1$ spinor Bose gas are briefly stated. In Section 3, we analyze the phase transition as the temperature reaches the first transition temperature. In Section 4, we analyze the behaviors before the second phase transition. On reaching the second transition temperature, we find that the numbers of thermal atoms in each component are equal to each other. In Section 5, we show the results of ideal spinor Bose gas of arbitrary F. Finally, some concluding remarks are given in Section 6.

2 The distribution of the spinor Bose gas

Consider the system of hyperfine spin $F = 1$ and $M_F = +1, 0, -1$. The total number of atoms N is the sum over the atoms in the three components

$$N = N_+ + N_0 + N_-, \tag{1}$$

and the net magnetization is

$$M = N_+ - N_-. \tag{2}$$

M and N are constants and are two constraints of the system. In current experiments, the range of N lies from hundreds to millions. This number is not truly macroscopic and can be called *mesoscopic*. As a consequence, the thermodynamic limit ($N \longrightarrow \infty$) has never been reached exactly. Hence, strictly speaking, BEC of the trapped gases is not a phase transition. In practice, the macroscopic occupation of the lowest state changes rather abruptly as the temperature lowered and can be observed. Thus, the terminology *phase transition* is generally used.

2.1 The grand potential

We introduce two Lagrange multipliers μ and η for the constraints N and M. The grand potential [16] for the general $F = 1$ spinor system is given by

$$\Omega = E - TS - \mu N - \eta M \tag{3}$$

where

$$E = \int d^3r \left\{ \frac{\hbar^2}{2m} \left|\nabla \mathbf{\Phi}\right|^2 + V(\mathbf{r}) \left|\mathbf{\Phi}\right|^2 + \frac{1}{2} g_n \left|\mathbf{\Phi}\right|^4 + \frac{1}{2} g_s \left|\mathbf{\Phi}^\dagger \mathbf{F} \mathbf{\Phi}\right|^2 \right\}. \tag{4}$$

Here $V(\mathbf{r})$ is the trapping potential. The coupling constants g_n and g_s characterizing the density-density and spin-spin interaction are given by $g_n = 4\pi\hbar^2(a_0+2a_2)/3m$ and $g_s = 4\pi\hbar^2(a_2 - a_0)/3m$. a_0 and a_2 are the corresponding scattering lengths of two spin-1 atoms collides into total spin-0, and spin-2 channel, respectively. Unlike that of the single-component BEC, the order parameter $\mathbf{\Phi}$ for this spinor system is vector-like whose components are represented by three classical fields, $\mathbf{\Phi}_+, \mathbf{\Phi}_0$ and $\mathbf{\Phi}_-$, corresponding to the condensates in the hyperfine states $|F = 1, M_F = +1, 0, -1\rangle$, (designated as state index $\alpha = +, 0, -$; hereafter), respectively. $\mathbf{F}$ is the angular momentum operator of $F = 1$. With the conventions of equations (1) and (2), the normalization of each order parameter is given by

$$\int d^3r \left|\mathbf{\Phi}_\alpha\right|^2 = 1. \tag{5}$$

The trapping potential is generally described as $V = m(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2)/2$. At thermal equilibrium, the system is in the minimum of grand potential. Let $n_{\alpha,j}$ be the occupation number of atoms at an energy level ϵ_j of the spinor component α. The ensemble averaged number $n_{\alpha,j}$ is then given by $\delta\Omega/\delta n_{\alpha,j} = 0$, which yields the standard result [11]

$$n_{\alpha,j} = \frac{z_\alpha \exp(-\epsilon_j/k_B T)}{1 - z_\alpha \exp(-\epsilon_j/k_B T)} \tag{6}$$

where $z_\alpha = \exp(\mu_\alpha/k_B T)$ is the fugacity. Mathematically we can put η into the chemical potential, and define the chemical potential for each spinor component as $\mu_\pm = \mu \pm \eta$ and $\mu_0 = \mu$.

To calculate the thermodynamic quantities, we use the grand canonical ensemble. This formulation corresponds to an open system, not an isolated system. The numbers N and M are ensemble averaged fixed. Only the grand canonical ensemble has analytic formulation and the error is negligible. For $N > 10^2$, the error is smaller than 10% [17].

2.2 The effective energy spectra

In the following, we consider the case of ideal spinor Bose gas. The scattering lengths a_0 and a_2 are set to zero. The system becomes the noninteracting vector bosons in a harmonic trap. We absorb the parameter η into the energy level mathematically. Then the effective energy spectra are

$$\begin{cases} \epsilon_+ = \epsilon^{ide} - \eta \\ \epsilon_0 = \epsilon^{ide} \\ \epsilon_- = \epsilon^{ide} + \eta \end{cases}. \tag{7}$$

With the 3-D harmonic trap, $\epsilon^{ide} = \sum_{i=1}^{3}(1/2+n_i)\hbar\omega_i$ are the energy levels. For $k_B T \gg \hbar\omega$, the number of thermal atoms in each component α, is thus found to be

$$N_\alpha^T = \sum_{\{\epsilon_j\}} n_{\alpha,j} = \left(\frac{k_B T}{\hbar\omega}\right)^3 g_3(z_\alpha), \tag{8}$$

where $g_\nu(x) = \sum_{j=1}^{\infty}(x^j/j^\nu)$ is the Bose-Einstein function, and $\omega = (\omega_1\omega_2\omega_3)^{1/3}$ is the geometrical mean of the trap frequencies. Typical number of N_α^T is $O(10^4) \sim O(10^7)$. Notice the formula (Eq. (8)) is truncated to $O(N^{1/3})$ and is exact up to $O(N^{1/2})$.

3 The first phase transition

3.1 The thermodynamic limit

As the temperature drops down to certain ultracold region, the first phase transition will occur. Assume $M > 0$, then η is positive to minimize Ω. (The case of $M < 0$ can be derived similarly.) Near transition temperature, $\mu \to \epsilon_+^{c,ide} = \epsilon_0^{ide} - \eta$, where $\epsilon_0^{ide} = 3\hbar\bar{\omega}/2 = \hbar(\omega_1 + \omega_2 + \omega_3)/2$. We define the effective fugacity $z_e = \exp[(\mu - \epsilon_+^{c,ide})/k_BT] = z_+ \exp(-\epsilon_0^{ide}/k_BT)$, where $z_+ = \exp[(\mu+\eta)/k_BT]$. Notice that when $|+1\rangle$ condensate starts, we have $z_e \to 1$ instead of $z_+ \to 1$.

The following approximation will be useful in the following derivations for $z \to 1$:

$$g_3(az) = \sum_{j=1}^{\infty} \frac{a^j z^j}{j^3} = \sum_{j=1}^{\infty} \frac{a^j (1+z-1)^j}{j^3} \simeq \sum_{j=1}^{\infty} \frac{a^j - j(1-z)a^j}{j^3} = g_3(a) - (1-z)g_2(a). \quad (9)$$

With $\exp(\hbar\bar{\omega}/k_BT) \to 1$, apply previous formula, the Bose distributions become

$$N_+^T = \left(\frac{k_BT}{\hbar\omega}\right)^3 g_3(z_e) + \frac{3\bar{\omega}}{2\omega}\left(\frac{k_BT}{\hbar\omega}\right)^2 g_2(z_e),$$
$$N_0^T = \left(\frac{k_BT}{\hbar\omega}\right)^3 g_3(z_e e^{-\beta\eta}) + \frac{3\bar{\omega}}{2\omega}\left(\frac{k_BT}{\hbar\omega}\right)^2 g_2(z_e e^{-\beta\eta}),$$
$$N_-^T = \left(\frac{k_BT}{\hbar\omega}\right)^3 g_3(z_e e^{-2\beta\eta}) + \frac{3\bar{\omega}}{2\omega}\left(\frac{k_BT}{\hbar\omega}\right)^2 g_2(z_e e^{-2\beta\eta}), \quad (10)$$

where $\beta = 1/k_BT$. We can see that $N_+ > N_0 > N_-$. When the first condensation occurs, we have $z_e \to 1$, then

$$N_+^T = \left(\frac{k_BT}{\hbar\omega}\right)^3 [g_3(1) - (1-z_e)g_2(1)] + \frac{3\bar{\omega}}{2\omega}\left(\frac{k_BT}{\hbar\omega}\right)^2 g_2(1),$$
$$N_0^T = \left(\frac{k_BT}{\hbar\omega}\right)^3 [g_3(e^{-\beta\eta}) - (1-z_e)g_2(e^{-\beta\eta})] + \frac{3\bar{\omega}}{2\omega}\left(\frac{k_BT}{\hbar\omega}\right)^2 g_2(e^{-\beta\eta}),$$
$$N_-^T = \left(\frac{k_BT}{\hbar\omega}\right)^3 [g_3(e^{-2\beta\eta}) - (1-z_e)g_2(e^{-2\beta\eta})] + \frac{3\bar{\omega}}{2\omega}\left(\frac{k_BT}{\hbar\omega}\right)^2 g_2(e^{-2\beta\eta}). \quad (11)$$

The first dominant term in each component is of $O(N)$, so we decide the thermodynamic limit transition temperature T_1^0 and z_η from the following two relationships for given N and M:

$$N = \left(\frac{k_BT_1^0}{\hbar\omega}\right)^3 [g_3(1) + g_3(z_\eta) + g_3(z_\eta^2)],$$
$$M = \left(\frac{k_BT_1^0}{\hbar\omega}\right)^3 [g_3(1) - g_3(z_\eta^2)], \quad (12)$$

where $z_\eta = e^{-\beta\eta}$. For convenience, we define

$$G_\nu^F(a) = g_\nu(1) + g_\nu(a) + g_\nu(a^2) + \cdots + g_\nu(a^{2F}), \quad (13)$$

then the number of atoms in each component is

$$N_+^T = \frac{g_3(1)}{G_3^1(z_\eta)}N,$$
$$N_0^T = \frac{g_3(z_\eta)}{G_3^1(z_\eta)}N,$$
$$N_-^T = \frac{g_3(z_\eta^2)}{G_3^1(z_\eta)}N. \quad (14)$$

The transition temperature is found as

$$T_1^0 = \frac{\hbar\omega}{k_B}\left(\frac{N}{G_3^1(z_\eta)}\right)^{1/3}. \quad (15)$$

The transition temperature also depends on M implicitly through z_η. If $M = N$ then from equation (12), we have $z_\eta \to 0$. It reduces to the ideal Bose gas of the usual one-component case. As a special case for $M = 0$ in spinor Bose gas, equation (12) gives $z_\eta = 1$, or $\eta = 0$, then

$$N = 3\zeta(3)\left(\frac{k_BT_1^0}{\hbar\omega}\right)^3,$$

where $g_\nu(1) = \zeta(\nu)$ is the Riemann's Zeta function. The transition temperature has the following simplified form

$$T_c^0 = \frac{\hbar\omega}{k_B}\left(\frac{N}{3\zeta(3)}\right)^{1/3} \simeq 0.652\frac{\hbar\omega}{k_B}N^{1/3}, \quad (16)$$

where $\zeta(3) = 1.202$.

3.2 The effect of finite particle number

Next, we discuss the correction to the previous first transition temperature of the spinor Bose gas in the thermodynamic limit. We will treat the effect of finite particle number first. The second dominant terms in equation (11) are in $O(N^{2/3})$. It will be used to decide the finite-size effect transition temperature T_1. Follow Pathria [15], neglecting $(1 - z_e)$ terms in equation (11), because they are in the order of $O(N^{1/2})$. At the moment of transition, the total number of atoms is the sum of two leading order terms, then

$$N = \left(\frac{k_BT_1}{\hbar\omega}\right)^3 G_3^1(z_\eta) + \frac{3\bar{\omega}}{2\omega}\left(\frac{k_BT}{\hbar\omega}\right)^2 G_2^1(z_\eta). \quad (17)$$

Let $T_1 = T_1^0(1 + \Delta t_1^{fin})$, we obtain the fractional correction on transition temperature due to the effect of finite particle number as

$$\Delta t_1^{fin} = -\frac{\bar{\omega}}{2\omega}\left(\frac{\hbar\omega}{k_B T_1^0}\right)\frac{G_2^1(z_\eta)}{G_3^1(z_\eta)}. \tag{18}$$

Specially for $M = 0$, we have

$$\Delta t_1^{fin} = -\frac{\bar{\omega}}{2\omega}\left(\frac{\hbar\omega}{k_B T_1^0}\right)\frac{\zeta(2)}{\zeta(3)} = -1.049\frac{\bar{\omega}}{\omega}N^{-1/3}, \tag{19}$$

where $\zeta(2) = 1.6449$.

3.3 The populations of condensate states

In equation (11), the lowest order terms are in $O(N^{1/2})$, summing up to give the number of condensate atoms

$$N^c = \left(\frac{k_B T_1}{\hbar\omega}\right)^3 G_2^1(z_\eta)(1 - z_e), \tag{20}$$

so we derive the number of condensate atom N_+^c from

$$N_+^c = \frac{1}{1 - z_e} = \left[\frac{N G_2^1(z_\eta)}{G_3^1(z_\eta)}\right]^{1/2}. \tag{21}$$

And the numbers of atoms in the other two condensates are

$$N_0^c = \frac{1}{1 - z_e z_\eta} \simeq \frac{k_B T_1^0}{\eta},$$
$$N_-^c = \frac{1}{1 - z_e z_\eta^2} \simeq \frac{k_B T_1^0}{2\eta}. \tag{22}$$

These N_0^c and N_-^c are in order of $\sim O(1)$ and are much smaller than N_+^c.

4 The second phase transition

4.1 The properties for temperature lies between the two phase transitions

With $\eta > 0$, we have

$$N_+^T > N_0^T > N_-^T,$$
$$N_+^c > N_0^c > N_-^c. \tag{23}$$

As the temperature drops below the first transition temperature T_1, the numbers of thermal atoms are

$$N_+^T = \left(\frac{k_B T}{\hbar\omega}\right)^3 g_3(1) + \frac{3\bar{\omega}}{2\omega}\left(\frac{k_B T}{\hbar\omega}\right)^2 g_2(1),$$

$$N_0^T = \left(\frac{k_B T}{\hbar\omega}\right)^3 g_3(z_\eta) + \frac{3\bar{\omega}}{2\omega}\left(\frac{k_B T}{\hbar\omega}\right)^2 g_2(z_\eta),$$

$$N_-^T = \left(\frac{k_B T}{\hbar\omega}\right)^3 g_3(z_\eta^2) + \frac{3\bar{\omega}}{2\omega}\left(\frac{k_B T}{\hbar\omega}\right)^2 g_2(z_\eta^2). \tag{24}$$

The following properties can be justified.

Lemma a: *N_+^T decreases monotonically as temperature drops.*

It is straightforward from formula N_+^T in equation (24).

Lemma b: *N_+^c increases monotonically as temperature drops.*

From Lemma a, as temperature decreases, the number of thermal atoms in component $|+\rangle$ will go to N_+^c or to N_0^T, N_-^T. The migration to N_0^c and N_-^c is negligible due to the relative small order. As N_+^T decreases and becomes equal to N_0^T, N_-^T has also increased, too. Since $M = N_+^T + N_+^c - N_-^T$, to keep the conservation of M, N_+^c will keep increasing as temperature drops.

Lemma c: *z_η increases monotonically as temperature drops.*

Let the total number of thermal atom be N^T and the magnetization of thermal atoms be M^T, from equation (16), we have

$$\frac{M^T}{N^T} = \frac{M - N_+^c}{N - N_+^c} = \frac{g_3(1) - g_3(z_\eta^2)}{G_3^1(z_\eta)}. \tag{25}$$

Because M and N are fixed and N_+^c is increasing as temperature drops, so the ratio is decreasing. Since g_3 and G_3^1 are monotonic increasing functions, z_η must be increasing to satisfy the property.

Lemma d: *The upper bound of z_η is 1.*

Since z_η is increasing as T decreases, while $z_\eta > 1$ implies $N_- > N_+$. This contradicts our condition $M > 0$.

4.2 The thermodynamic limit

From the properties derived above, we know the upper bound z_η is 1. The second phase transition occurs around $z_\eta \sim 1$. Expand equation (11) around $z_\eta = 1$, we get

$$N_+^T = \left(\frac{k_B T}{\hbar\omega}\right)^3 [g_3(1) - (1 - z_e)g_2(1)] + \frac{3\bar{\omega}}{2\omega}\left(\frac{k_B T}{\hbar\omega}\right)^2 g_2(1),$$
$$N_0^T = \left(\frac{k_B T}{\hbar\omega}\right)^3 [g_3(1) - (1 - z_e z_\eta)g_2(1)] + \frac{3\bar{\omega}}{2\omega}\left(\frac{k_B T}{\hbar\omega}\right)^2 g_2(1),$$
$$N_-^T = \left(\frac{k_B T}{\hbar\omega}\right)^3 [g_3(1) - (1 - z_e z_\eta^2)g_2(1)] + \frac{3\bar{\omega}}{2\omega}\left(\frac{k_B T}{\hbar\omega}\right)^2 g_2(1). \tag{26}$$

The leading order terms in the above formulas are in $O(N)$ and *are equal to each other for the three components.* The second phase transition temperature T_2^0 in the thermodynamic limit is defined from these terms

$$N_+^T = N_0 = N_- = \frac{N-M}{3} = \left(\frac{k_B T_2^0}{\hbar\omega}\right)^3 g_3(1).$$

Notice that at T_2^0, all thermal atom numbers in the three components are equal to each other, and the dominant condensate atoms are in $|+\rangle$ state only. We have

$$M = N_+ - N_- = N_+^c. \quad (27)$$

So, the thermodynamic limit second transition temperature is

$$T_2^0 = \frac{\hbar\omega}{k_B}\left[\frac{N-M}{3g_3(1)}\right]^{1/3} = 0.652\frac{\hbar\omega}{k_B}(N-M)^{1/3}. \quad (28)$$

4.3 The effect of finite particle number

From the Bose distribution, we obtain the number of condensate atoms in each component

$$N_+^c = \frac{1}{1-z_e} = M,$$
$$N_0^c = \frac{1}{1-z_e z_\eta},$$
$$N_-^c = \frac{1}{1-z_e z_\eta^2}. \quad (29)$$

The first term in equation (26) is of $O(N)$, the second term is of $O(N^{2/3})$ in all components. Near the second transition temperature, the numbers of condensate atoms can be neglected. We decide the second transition temperature T_2 of finite particle number by the two leading terms in equation (26)

$$\frac{N-M}{3} = \left(\frac{k_B T_2}{\hbar\omega}\right)^3 g_3(1) + \frac{3\bar{\omega}}{2\omega}\left(\frac{k_B T_2}{\hbar\omega}\right)^2 g_2(1). \quad (30)$$

Let $T_2 = T_2^0(1+\Delta t_2^{fin})$, then the fractional modification of transition temperature due to the finite particle number is

$$\Delta t_2^{fin} = -\frac{\bar{\omega}}{2\omega}\left(\frac{\hbar\omega}{k_B T_2^0}\right)\frac{g_2(1)}{g_3(1)} = -1.049\frac{\bar{\omega}}{\omega}(N-M)^{-1/3}. \quad (31)$$

4.4 The number of condensate atoms in each component

The condensation number N_+^c is

$$N_+^c = M = \frac{1}{1-z_+} = \frac{1}{1-e^{\beta(\mu-\epsilon_+^c)}} \simeq \frac{k_B T_2^0}{\epsilon_0 - \eta - \mu} \quad (32)$$

or

$$\mu = \epsilon_0^{ide} - \eta - \frac{k_B T}{M}. \quad (33)$$

For the other two condensate numbers,

$$N_0^c = \frac{1}{1-e^{\beta(\mu-\epsilon_0^c)}} = \frac{k_B T_2}{\eta + \frac{k_B T}{M}} \simeq \frac{k_B T_2}{\eta} = O(N^{1/2}), \quad (34)$$

where $\eta/(\hbar\omega) = O(N^{-1/6})$ and $\eta/(k_B T_2) = O(N^{-1/2}) \gg 1/M = O(N^{-1})$, and $1/M$ can then be neglected; so

$$N_-^c = \frac{k_B T_2}{2\eta}. \quad (35)$$

Combine the expressions of N_0^c and N_-^c with that from equation (26)

$$N_0^c + N_-^c = \left(\frac{k_B T}{\hbar\omega}\right)^3 g_2(1)[(1-z_0)+(1-z_-)], \quad (36)$$

we obtain

$$\frac{3k_B T_2}{2\eta} = \frac{N-M}{3}\frac{g_2(1)}{g_3(1)}\left[\frac{3\eta}{k_B T_2}\right]. \quad (37)$$

Finally, we obtain condensate number of components $|0\rangle$ and $|-\rangle$ as

$$N_0^c = \frac{k_B T_2}{\eta} = \left(\frac{2(N-M)g_2(1)}{3g_3(1)}\right)^{1/2} = 0.955(N-M)^{1/2},$$
$$N_-^c = \frac{k_B T_2}{2\eta} = \left(\frac{(N-M)g_2(1)}{6g_3(1)}\right)^{1/2} = 0.478(N-M)^{1/2}. \quad (38)$$

For temperature below T_2, the condensate number of component $|\alpha\rangle$ and the total number of condensate atoms satisfy

$$N^c = \sum_\alpha N_\alpha^c = (N-M)\left[1-\left(\frac{T}{T_2}\right)^3\right] + M, \quad (39a)$$
$$M = \sum_\alpha \alpha N_\alpha^c = N_+^c - N_-^c, \quad (39b)$$
$$\frac{1}{N_-^c} = \frac{2}{N_0^c} - \frac{1}{N_+^c}. \quad (39c)$$

5 Arbitrary F ideal spinor Bose gas

5.1 The first phase transition

Following the previous analysis, we can easily derive the first phase transition of arbitrary F ideal spinor Bose gas:

$$N = \left(\frac{k_B T_1^0}{\hbar\omega}\right)^3 G_3^F(z_\eta),$$
$$M = \left(\frac{k_B T_1^0}{\hbar\omega}\right)^3 \sum_{i=0}^{2F}(F-i)g_3(z_\eta^i). \quad (40)$$

The thermodynamic limit first transition temperature is

$$T_1^0 = \frac{\hbar\omega}{k_B}\left(\frac{N}{G_3^F(z_\eta)}\right)^{1/3}, \tag{41}$$

and the shift of transition temperature due to finite particle number is

$$\begin{aligned}\Delta t_1^{fin} &= -\frac{\bar{\omega}}{2\omega}\left(\frac{\hbar\omega}{k_B T_1^0}\right)\frac{G_2^F(z_\eta)}{G_3^F(z_\eta)}\\ &= -\frac{\bar{\omega}}{2\omega}\frac{G_2^F(z_\eta)}{[G_3^F(z_\eta)]^{2/3}}N^{-1/3}, \end{aligned}\tag{42}$$

and the condensate population of $|F, m_F = F\rangle$ is

$$N_F^c = \left[\frac{NG_2^F(z_\eta)}{G_3^F(z_\eta)}\right]^{1/2}. \tag{43}$$

5.2 The second phase transition

For arbitrary F ideal spinor Bose gas, there are two and only two phase transition. The thermodynamic limit second transition temperature is

$$T_2^0 = \frac{\hbar\omega}{k_B}\left[\frac{N-M}{(2F+1)g_3(1)}\right]^{1/3} = 0.940\frac{\hbar\omega}{k_B}\left[\frac{N-M}{(2F+1)}\right]^{1/3}. \tag{44}$$

The shift of transition temperature due to finite particle number is

$$\Delta t_2^{fin} = -\frac{\bar{\omega}}{2\omega}\left(\frac{\hbar\omega}{k_B T_2^0}\right)\frac{g_2(1)}{g_3(1)} = -0.728\frac{\bar{\omega}}{\omega}\left(\frac{2F+1}{N-M}\right)^{1/3}. \tag{45}$$

Define the average of the harmonic sequences from 1 to $2F$ as

$$\bar{H}_{2F} \equiv \frac{1}{2F}\sum_{n=1}^{2F}\frac{1}{n}, \tag{46}$$

we obtain

$$\frac{k_B T_2^0}{\eta} = \left[\frac{(N-M)}{2\bar{H}_{2F}}\right]^{1/2}. \tag{47}$$

The condensate population of $|F, m_F = +F\rangle$ is equal to M. For the other components $|F, m_F = \alpha\rangle$

$$N_\alpha^c = \frac{1}{F-\alpha}\left(\frac{(N-M)}{2\bar{H}_{2F}}\right)^{1/2}, \quad (\alpha = -F, -F+1, \cdots, F-1). \tag{48}$$

For temperature below T_2, the condensate population of component $|\alpha\rangle$ and the total number of condensate atoms satisfy

$$N^c = \sum_\alpha N_\alpha^c = (N-M)\left[1-\left(\frac{T}{T_2}\right)^3\right] + M \tag{49a}$$

$$M = \sum_\alpha \alpha N_\alpha^c, \tag{49b}$$

$$\frac{1}{N_\alpha^c} = \frac{F-\alpha}{N_{F-1}^c} + \frac{1-(F-\alpha)}{N_F^c}. \tag{49c}$$

6 Discussion

1. We analyzed the thermodynamic properties of the trapped ideal spinor Bose gas. Without loss of physics, we discussed the case of $M > 0$ (the case of $M < 0$ can be derived in the same way). As temperature drops to T_1, the condensate atoms start to appear in the highest M_F spinor component. As the temperature keeps dropping to T_2, the numbers of thermal atoms in each spinor component become equal to each other and the condensate atoms in the other spinor components start to appear. Thus, there are two phase transitions only.

2. Our calculation is exact up to $O(N^{1/2})$. Typically, if N_α^T in each component is larger than 10^3, then $k_B T/(\hbar\omega) > 10$, the approximation of equation (9) is reliable throughout our analysis.

3. Under a weak applied magnetic field B along z-direction, the Larmor frequency is $\omega_L = \gamma_\mu B$, where γ_μ is the gyromagnetic ratio. The energy shift to each component is $-M_F\hbar\omega_L = -M_F\eta_{ext}$. Combine the shift due to the applied magnetic field together with the Lagrange multiplier η, we have the effective magnetic field,

$$\eta_{eff} = \eta + \eta_{ext}. \tag{50}$$

There will be no change in our analysis with η being replaced by η_{eff}. The parameter η_{eff} plays the role of an effective magnetic field, As temperature drops to T_2, $\eta_{eff} \to 0$. The effective magnetic field disappears thereafter.

4. In this system, we have two conservation conditions and have two Lagrange multipliers. Whenever there is a discontinuity in the temperature derivative of the Lagrange multiplier, there is a phase transition.

5. The two phases on either side of the critical temperature have different symmetries and must be described by different functions [21]. Landau proposed the *order parameter* to describe the low-temperature phase, which is usually an extensive thermodynamic variable accessible to measurements. In the spinor BEC system, near the first transition temperature T_1, we can derive $1/N_+^c \sim -\beta(\mu_+ - \epsilon_0^{ide})$ from equation (21). Thus, for temperature lies above T_1, we can use $\beta(\mu_+ - \epsilon_0^{ide})$ to describe the incoherence of the system. We may call it the *disorder parameter.* As temperature decreases to the phase transition, the disorder parameter disappears and there is symmetry broken. The corresponding order parameter shows up. The condensate fraction f_α is commonly used for the measurable quantity. Similar picture can be applied to the behaviors near the second phase transition temperature T_2. We plot in Figure 1 the order and disorder parameters as function of temperature near the phase transitions for the case of $N = 10^5$ and $M = 6\times10^4$. Furthermore, Figure 2a shows details near the first phase transition. Due to the finite-size effect and trap potential, the phase transition temperature T_1 shifts downward to the thermodynamic T_1^0 and the curve of order and disorder parameter henceforth has a separation at T_1. Figure 2b plots similar behaviors of the second phase transition in more details, too.

6. The Double phase transition phenomena base on exact ideal spinor Bose gas. In the interacting spinor Bose

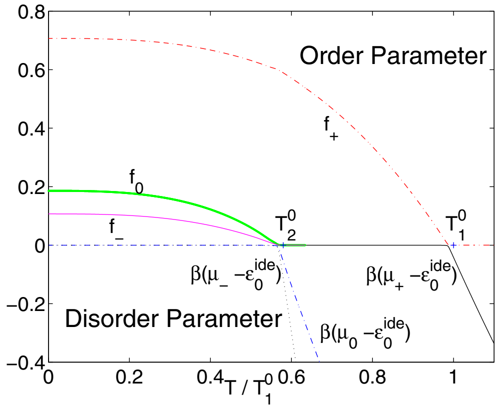

Fig. 1. The order parameter and disorder parameter of the spinor BEC system. The two kinds of parameters describe the two phases on each side of the critical temperature. f_α denotes the condensate fraction of spinor state $|\alpha\rangle$.

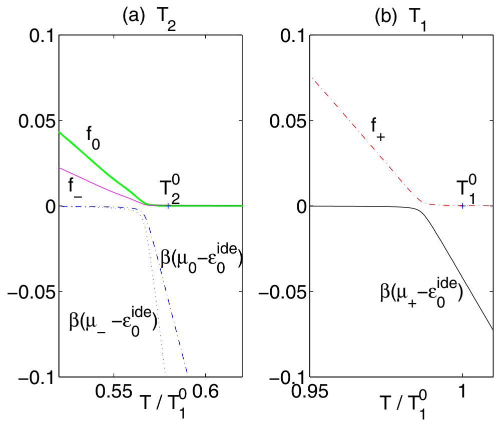

Fig. 2. (a) The detailed behaviors of order and disorder parameters near the first transition temperature T_1; (b) the behaviors near the second phase transition temperature T_2.

gas, there are $2F$'s parameters to describe the behaviors of the phase transition. The phase transitions can be more than two. A systematic discussion of the thermodynamic properties of the weakly interacting trapped spinor $F = 1$ Bose gas, will be the subject of a future work.

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